Lie symmetries of radiation natural convection flow past an inclined surface

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Abstract

Applying Lie symmetry method, we find the most general Lie point symmetry groups of the radiation natural convection flow equation(RNC). Looking the adjoint representation of the obtained symmetry group on its Lie algebra, we will find the preliminary classification of its group-invariant solutions.

Keywords: Lie-point symmetries, invariant, optimal system of Lie subalgebras.

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1 Introduction

This paper can be viewed as a continuation of the work of paper [7] where the authors by using Lie group analysis for the PDE system corresponding to radiation effects on natural convection heat transfer past an inclined surface (RNC) could reduce this system to an ODE one and obtain numerical solutions by applying Runge-Kutta method.

There are various techniques for finding solutions of differential equations but most of them are useful for a few classes of equations and applying these techniques for unknown equations is impossible. Fortunately symmetries of differential equations remove this problem and give exact solutions.

Symmetry methods for differential equations were discussed by S.Lie at first. One of the most important Lie surveys was finding relationships between continuous transformation group and their infinitesimal generators that allows us to reduce invariant conditions of differential equations

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under group action, that are complicated because of non linearity, to linear ones.

The main results of applying Lie symmetry method for differential equations is finding group invariant solutions. A useful way for reducing equations is finding any subgroup of the symmetry group and writing invariant solutions with respect to this subalgebra. This reduced equation is of fewer variables and is easier to solve. In fact for many important equations arising from geometry and physics these invariant solutions are the only ones which can be studied thoroughly.

This type of equations as (RNC) system plays an important role in engineering and industrial fields. In this paper Lie symmetry method is applied to find the most general Lie symmetry group and optimal system of (RNC). Finally, we gain group-invariant solutions of the reduced system.

2 Symmetries of RNC System

Consider the (RNC) system:

$$
\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}+Gr\theta\cos\alpha,\\
&u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{Pr}(1+4R)\frac{\partial^2 \theta}{\partial y^2}.
\end{aligned}
\tag{1}
$$

where $Gr=\dfrac{g\beta(T_W-T_\infty)\nu}{U_\infty^3}$ is the Grashof number, $Pr=\dfrac{\rho c_p\nu}{k}$ is the Prandtl number and $R=\dfrac{4\sigma_0 T_\infty^3}{3k^8}$ is the radiation parameter [7]. Now the infinitesimal method is implied as follow:

The infinitesimal generator $\mathbf{X}$ on total space of the form:

$$
\mathbf{X}=\sum_{i=1}^{p}\xi^i(x,u)\partial_{x^i}+\sum_{j=1}^{q}\eta^j(x,u)\partial_{u^j}, \tag{2}
$$

has the n^{th} order prolongation $\mathrm{Pr}^{(n)}X$ ([6], Th 4.16). Applying the fundamental infinitesimal symmetry criterion ([6], Th 6.5) on X as follow:

$$
\mathrm{Pr}^{(n)}\mathbf{X}\big[\Delta_\nu(x,u^{(n)})\big]=0,\quad \nu=1,\cdots,l,\quad \text{whenever}\quad \Delta_\nu(x,u^{(n)})=0. \tag{3}
$$

obtains the infinitesimal generators of the symmetry group.

The vector field associated with RNC system is of the form:

$$
\begin{aligned}
X=&\ \xi_1(x,y,u,v)\partial_x+\xi_2(x,y,u,v)\partial_y\\
&+\varphi_1(x,y,u,v)\partial_u+\varphi_2(x,y,u,v)\partial_v+\varphi_3(x,y,u,v)\partial_t.
\end{aligned}
\tag{4}
$$

Since in RNC system, second order derivatives are appeared, symmetry generators are obtained by applying (3) for the second prolongation of X.

The vector field $\mathbf{X}$ generates a one parameter symmetry group of RNC $= (\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ if and only if equations (3) hold for $\nu = 1, 2, 3, 4$. So symmetry group of RNC system is spanned by following infinitesimal generators:

$$X_1 = \partial_x, \qquad X_2 = \partial_y,$$
$$X_3 = x\,\partial_x + u\partial_u + t\partial_t, \qquad X_4 = 2x\partial_x + y\partial_y - v\partial_v - 2t\partial_t.$$

The Lie table of Lie algebra $\mathfrak{g}$ for RNC system is given below:

	X_1	X_2	X_3	X_4
X_1	0	0	X_1	$2X_1$
X_2	0	0	0	X_2
X_3	$-X_1$	0	0	0
X_4	$-2X_1$	$-X_2$	0	0

Theorem 1. *If $g_k(h)$ be the one parameter group generated by X_k, $k = 1, \cdots, 4$, then*

$$\begin{array}{lllll} g_1 & : & (x, y, u, v, t) & \longmapsto & (h + x, y, u, v, t), \\ g_2 & : & (x, y, u, v, t) & \longmapsto & (x, h + y, u, v, t), \\ g_3 & : & (x, y, u, v, t) & \longmapsto & (xe^h, y, ue^h, v, te^h), \\ g_4 & : & (x, y, u, v, t) & \longmapsto & (xe^{2h}, ye^h, u, ve^{-h}, te^{-2h}). \end{array} \tag{5}$$

3 Optimal System

The aim of Lie theory is classifying the invariant solutions and reducing equations. Some of Lie algebras contain different subalgebras, so classifying them plays an important role in transforming equations into easier ones. Therefore we are classifying these subalgebras up to adjoint representation and finding an optimal system of subalgebras instead of finding an optimal system of subgroups.
The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon \mathrm{X_i})\mathrm{X_j}) = \mathrm{X_j} - \varepsilon[\mathrm{X_i}, \mathrm{X_j}] + \frac{\varepsilon^2}{2}[\mathrm{X_i}, [\mathrm{X_i}, \mathrm{X_j}]] - \cdots, \tag{6}$$

where $[X_i, X_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 4$ ([5], ch 3.3).

Theorem 2. *A one-dimensional optimal system of RNC results as follow:*

1) X_1,	4) $X_1 + X_2$,	7) $X_3 - X_2$,
2) X_2,	5) $X_2 - X_1$,	8) $X_3 + X_4$,
3) X_3,	6) $X_2 + X_3$,	9) $X_4 - X_3$.

Proof: Consider the symmetry algebra $\mathfrak{g}$ of the RNC system with adjoint representation demonstrated in the below table:

	X_1	X_2	X_3	X_4
X_1	$X_1+\varepsilon X_3+2\varepsilon X_4$	X_2	X_3	X_4
X_2	X_1	$X_2+\varepsilon X_4$	X_3	X_4
X_3	$e^{-\varepsilon}X_1$	X_2	X_3	X_4
X_4	$e^{-2\varepsilon}X_1$	$e^{-\varepsilon}X_2$	X_3	X_4

Let

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4. \tag{7}$$

In this stage our goal is to simplify as many of the coefficient a_i as possible.

The following results are obtained from adjoint application on X:

1. Let $a_4 \neq 0$. Scaling X if necessary, suppose that $a_4 = 1$. Then

$$X' = a_1X_1 + a_2X_2 + a_3X_3 + X_4. \tag{8}$$

If we act on X' with $\mathrm{Ad}(\exp(a_3X_3))$, the coefficient of X_1 can be vanished:

$$X'' = a_2X_2 + a_3X_3 + X_4. \tag{9}$$

By applying $\mathrm{Ad}(\exp(a_4X_4))$ on X'', we will obtain X''' as follow:

$$X''' = a_3X_3 + X_4. \tag{10}$$

1-a. If $a_3 \neq 0$, then the coefficient of X_3 can be -1 or 1. Therefore every one-dimensional subalgebra generated by X in this case is equivalent to $X_3 + X_4$, $X_4 - X_3$.

1-b. If $a_3 = 0$, then every one-dimensional subalgebra generated by X in this case is equivalent to X_4.

2. The remaining one-dimensional subalgebras are spanned by vector fields of the form X with $a_4 = 0$.

2-a. If $a_3 \neq 0$, let $a_3 = 1$. By acting of $\mathrm{Ad}(\exp(a_3X_3))$ on X, we will have

$$\hat{X} = a_2X_2 + X_3. \tag{11}$$

2-a-1. If $a_2 \neq 0$, then the coefficient of X_2 can be -1 or 1. So every one-dimensional subalgebra generated by X in this case is equivalent to $X_2 + X_3$, $X_3 - X_2$.

2-a-2. If $a_2 = 0$ then every one-dimensional subalgebra generated by X_3.

2-b. Let $a_4 = 0, a_3 = 0$.

2-b-1. If $a_2 \neq 0$, then we can make the coefficient of X_1 either -1 or 1 or 0. Every one-dimensional subalgebra generated by X is equivalent to $X_1 + X_2$, $X_2 - X_1$, X_2.

2-b-2. If $a_2 = 0$, then every one-dimensional subalgebra generated by X is equivalent to X_1. □

4 Invariant Solutions and Reduction

Now for finding invariant solutions using characteristic method [2]:

$$Q_\alpha \mid_{u=u(x,t)} \equiv X[u^\alpha - u^\alpha(x,t)] \mid_{u=u(x,t)} \quad \alpha = 1, \cdots, M \tag{12}$$

where M is the number of dependant variables. Now invariant solutions for RNC result as follow:

Consider $X_1 = \partial_x$. Applying characteristic method on X_1 obtains: $Q_u = -\frac{\partial}{\partial x}u\left(x,y\right), Q_v = -\frac{\partial}{\partial x}v\left(x,y\right), Q_t = -\frac{\partial}{\partial x}t\left(x,y\right)$. Solutions of this system are of the form: $u = F_1\left(y\right), v = F_1\left(y\right), t = F_1\left(y\right)$.

Consider $X_2 = \partial_y$. Applying characteristic method on X_2 obtains: $Q_u = -\frac{\partial}{\partial y}u\left(x,y\right), Q_v = -\frac{\partial}{\partial y}v\left(x,y\right), Q_t = -\frac{\partial}{\partial y}t\left(x,y\right)$. Solutions of this system are of the form: $u = F_1\left(x\right), v = F_1\left(x\right), t = F_1\left(x\right)$.

Consider $X_3 = x\partial_x + u\partial_u + t\partial_t$. Applying characteristic method on X_3 obtains: $Q_u = u\left(x,y\right) - x\frac{\partial}{\partial x}u\left(x,y\right), Q_v = -x\frac{\partial}{\partial x}v\left(x,y\right), Q_t = t\left(x,y\right) - x\frac{\partial}{\partial x}t\left(x,y\right)$.
Solutions of this system are of the form: $u = xF_1\left(y\right), v = F_1\left(y\right), t = xF_1\left(y\right)$.

Consider $X_4 = 2x\partial_x + y\partial_y - v\partial_v - 2t\partial_t$. Applying characteristic method on X_4 obtains: $Q_u = -2\,x\frac{\partial}{\partial x}u\left(x,y\right) - y\frac{\partial}{\partial y}u\left(x,y\right), Q_v = -v\left(x,y\right) - 2\,x\frac{\partial}{\partial x}v\left(x,y\right) - y\frac{\partial}{\partial y}v\left(x,y\right), Q_t = -2\,t\left(x,y\right) - 2\,x\frac{\partial}{\partial x}t\left(x,y\right) - y\frac{\partial}{\partial y}t\left(x,y\right)$.
Solutions of this system are of the form: $u = F_1\left(\frac{y}{\sqrt{x}}\right), v = F_1\left(\frac{y}{\sqrt{x}}\right)\frac{1}{\sqrt{x}}, t = F_1\left(\frac{y}{\sqrt{x}}\right)x^{-1}$.

Consider $X_1 + X_2 = \partial_x + \partial_y$. Applying characteristic method on $X_1 + X_2$ obtains: $Q_u = \frac{\partial}{\partial x}u\left(x,y\right) + \frac{\partial}{\partial y}u\left(x,y\right), Q_v = \frac{\partial}{\partial x}v\left(x,y\right) + \frac{\partial}{\partial y}v\left(x,y\right), Q_t = \frac{\partial}{\partial x}t\left(x,y\right) + \frac{\partial}{\partial y}t\left(x,y\right)$. Solutions of this system are of the form: $u = F_1\left(-x+y\right), v = F_1\left(-x+y\right), t = F_1\left(-x+y\right)$.

Consider $X_2 - X_1 = \partial_y - \partial_x$. Applying characteristic method on $X_2 - X_1$ obtains: $Q_u = \frac{\partial}{\partial x}u\left(x,y\right) - \frac{\partial}{\partial y}u\left(x,y\right), Q_v = \frac{\partial}{\partial x}v\left(x,y\right) - \frac{\partial}{\partial y}v\left(x,y\right), Q_t = \frac{\partial}{\partial x}t\left(x,y\right) - \frac{\partial}{\partial y}t\left(x,y\right)$. Solutions of this system are of the form: $u = F_1\left(x+y\right), v = F_1\left(x+y\right), t = F_1\left(x+y\right)$.

Consider $X_2 + X_3 = x\partial_x + \partial_y + u\partial_u + t\partial_t$. Applying characteristic method on $X_2 + X_3$ obtains: $Q_u = u\left(x,y\right) - x\frac{\partial}{\partial x}u\left(x,y\right) - \frac{\partial}{\partial y}u\left(x,y\right), Q_v = -x\frac{\partial}{\partial x}v\left(x,y\right) - \frac{\partial}{\partial y}v\left(x,y\right), Q_t = t\left(x,y\right) - x\frac{\partial}{\partial x}t\left(x,y\right) - \frac{\partial}{\partial y}t\left(x,y\right)$. Solutions of this system are of the form: $u = xF_1\left(-\ln\left(x\right)+y\right), v = F_1\left(-\ln\left(x\right)+y\right), t = xF_1\left(-\ln\left(x\right)+y\right)$.

Consider $X_3 - X_2 = x\partial_x - \partial_y + u\partial_u + t\partial_t$. Applying characteristic method on $X_3 - X_2$ obtains: $Q_u = u\left(x,y\right) - x\frac{\partial}{\partial x}u\left(x,y\right) + \frac{\partial}{\partial y}u\left(x,y\right), Q_v =$

$-x\frac{\partial}{\partial x}v\left(x,y\right)+\frac{\partial}{\partial y}v\left(x,y\right), Q_t = t\left(x,y\right)-x\frac{\partial}{\partial x}t\left(x,y\right)+\frac{\partial}{\partial y}t\left(x,y\right)$. Solutions of this system are of the form: $u = xF_1\left(\ln\left(x\right)+y\right), v = F_1\left(\ln\left(x\right)+y\right), t = xF_1\left(\ln\left(x\right)+y\right)$.

Consider $X_3 + X_4 = 3x\partial_x + y\partial_y + u\partial_u - v\partial_v - t\partial_t$. Applying characteristic method on $X_3 + X_4$ obtains: $Q_u = u\left(x,y\right) - 3\,x\frac{\partial}{\partial x}u\left(x,y\right) - y\frac{\partial}{\partial y}u\left(x,y\right), Q_v = -v\left(x,y\right) - 3\,x\frac{\partial}{\partial x}v\left(x,y\right) - y\frac{\partial}{\partial y}v\left(x,y\right), Q_t = -t\left(x,y\right) - 3\,x\frac{\partial}{\partial x}t\left(x,y\right) - y\frac{\partial}{\partial y}t\left(x,y\right)$. Solutions of this system are of the form: $u = F_1\left(\frac{y}{\sqrt[3]{x}}\right)\sqrt[3]{x}, v = F_1\left(\frac{y}{\sqrt[3]{x}}\right)\frac{1}{\sqrt[3]{x}}, t = F_1\left(\frac{y}{\sqrt[3]{x}}\right)\frac{1}{\sqrt[3]{x}}$.

Consider $X_4 - X_3 = x\partial_x + y\partial_y - u\partial_u - v\partial_v - 3t\partial_t$. Applying characteristic method on $X_4 - X_3$ obtains: $Q_u = -u\left(x,y\right) - x\frac{\partial}{\partial x}u\left(x,y\right) - y\frac{\partial}{\partial y}u\left(x,y\right), Q_v = -v\left(x,y\right) - x\frac{\partial}{\partial x}v\left(x,y\right) - y\frac{\partial}{\partial y}v\left(x,y\right), Q_t = -3\,t\left(x,y\right) - x\frac{\partial}{\partial x}t\left(x,y\right) - y\frac{\partial}{\partial y}t\left(x,y\right)$. Solutions of this system are of the form: $u = F_1\left(\frac{y}{x}\right)x^{-1}, v = F_1\left(\frac{y}{x}\right)x^{-1}, t = F_1\left(\frac{y}{x}\right)x^{-3}$.

5 Conclusion

In this paper by applying infinitesimal symmetry methods, we find optimal system and finally could reduce the RNC system and find invariant solutions.

References

[1] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer, New York, 1989.

[2] P. Hydon , *Symmetries Methods for Differential Equations, A Beginner's Guid*, Cambridge: Cambridge University Press.

[3] M. Nadjafikhah , *Classification of similarity solutions for inviscid burgers' equation*, Adv. appl. Clifford alg. 20 (2010), 71-77.

[4] M. Nadjafikhah, *Lie Symmetries of Inviscid Burgers Equation*, Adv. Appl. Clifford Alg., 19. (2009) 101-112.

[5] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1993.

[6] P.J. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, 1995.

[7] S. Sivasankaran, M. Bhuvaneswari, P. Kandaswamy and E.K. Ramasami, *Lie group analysis of radiation natural convection flow past an inclined surface*, Communications in Nonlinear Science and Numerical Simulation 13 (2008) 269-276.